Analysis of Global Fixed Income Returns Using Multilinear Tensor Algebra

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KEY FINDINGS

- We show that global fixed income returns naturally reside on multi-modal lattice data structures, referred to as tensors, which exhibit highly structured correlations across maturities and economies.

- We introduce a multilinear algebraic approach to the modeling of the global term structure underlying multiple interest rate curves, which allows us to express the covariance of global returns as a joint multilinear decomposition of the maturity-domain and country-domain covariances.

- This drastically reduces the number of parameters required to fully capture the global returns covariance structure, and makes it possible to devise rigorous and tractable global portfolio management strategies which are tailored to each of the data domains.

ABSTRACT

Global fixed income returns exhibit highly structured correlations across maturities and economies (data modes), and their modeling therefore requires analysis tools that are capable of directly capturing the inherent multi-way couplings present in such multimodal data. Yet, current analyses typically employ “flat-view” multivariate matrix models and their associated linear algebras; these are agnostic to the global data structure and can only describe local linear pairwise relationships between data entries. To address this issue, we first show that global fixed income returns naturally reside on multi-modal lattice data structures, referred to as tensors. This serves as a basis to introduce a multilinear algebraic approach, inherent to tensors, to the modeling of the global term structure underlying multiple interest rate curves. Owing to the enhanced flexibility of multilinear algebra, statistical descriptors, such as correlations, exist between tensor columns and rows (fibers), as opposed to between individual entries in standard matrix analysis. This allows us to express the covariance of global returns as a joint multilinear decomposition of the maturity-domain and country-domain covariances. This not only drastically reduces the number of parameters required to fully capture the global returns covariance structure, but also makes it possible to devise rigorous and tractable global portfolio management strategies; we tailor these specifically to each of the data domains and thereby fully exploit the lattice structure of global fixed income returns. The ability of the proposed multilinear tensor approach to compactly describe the macroeconomic environment through economically meaningful factors is validated via empirical analysis that demonstrates the existence of maturity-domain and country-domain covariances underlying the interest rate curves of eight developed economies.

TOPICS

Fixed income and structured finance, quantitative methods, statistical methods, portfolio construction, global markets*
Participants in economic markets have long recognized the importance of identifying common factors (often of a hidden nature) in financial data. These co-movements then can be used to explain the returns of securities within an asset class. In such a task, it is critical to distinguish the common risks, which have a general impact on the returns of most securities, from the idiosyncratic risks, which influence securities individually. To this end, following from the seminal work in Litterman and Scheinkmann (1991), a significant portion of the fixed income literature has been devoted to the technique of principal component analysis (PCA) (Jolliffe 1986), which provides a parsimonious interpretation of the term structure dynamics of interest rate curves through latent factors, called the principal components (PCs). Empirical results suggest that the first three latent factors, referred to as:

1. **Level** – the parallel shift of the yield curve,
2. **Slope** – the tilting of the yield curve, and
3. **Curvature** – the flexing of the yield curve

are sufficient to almost fully reflect the behavior of the entire term structure. Moreover, the principal components in data covariance matrixes also frequently admit economically meaningful interpretation. This has made PCA a fundamental tool for characterizing single-economy interest rate curves. Overall, notable benefits of PCA include its:

- Analyticity and mathematical tractability, since it is formulated through classical linear algebra techniques
- Ability to parsimoniously describe economic factors in terms of the level, slope, and curvature components
- Application to stress-testing and scenario analysis using eigenvalue and eigenvector perturbation methods
- Direct applicability to portfolio hedging, owing to the orthogonality between principal components.

Despite the advantages of using PCA for domestic (single-country) fixed-income investing, financial institutions routinely invest globally where the increasingly growing interconnectedness of the international debt and money markets presents a major challenge for PCA-based risk management. This is largely due to the legacy PCA-based analytics that employ “flat-view” linear algebra methods, whereby trades are typically hedged by offsetting their domestic-curve principal components. The high correlation between interest rates across both maturities and countries leaves such strategies with limited avenues for diversification, and unprepared for cross-country co-movements arising from global macroeconomic events. The most recent credit crisis in 2007–2008, for example, shows how macroeconomic shocks can be transmitted across multiple interest rate curves. This all suggests that a parsimonious model for describing the co-movement of interest rates at the relevant maturities and in the relevant countries is necessary for global fixed income investors to adequately identify and manage risk.

These requirements have spurred modifications and variations of PCA for joint term structure analysis, with a variety of existing solutions largely stemming from the ambiguity in the problem formulation. Indeed, problem set-ups tend to differ widely with regard to the data processing, estimation procedure, number of latent factors required to explain the joint dynamics of multiple yield curves, and economic interpretation of the global and domestic factors obtained. For example, approaches that apply PCA to vectorized data obtained from multiple interest rate curves (Rodrigues 1997; Driessen 1992)...
et al. 2003; Novosyolov and Satchkov 2008) completely ignore the multi-curve lattice structure inherent to global interest rate curves, shown in Exhibit 1, and therefore yield factors that may be difficult to interpret and still tend to reflect local idiosyncratic and domestic behaviors. Furthermore, the numerical instability issues associated with covariance estimation lead to a counter-intuitive result, whereby the more collinear the asset returns, the greater the need for diversification. The corresponding increase in the number of assets, \( N \), to compensate for such ill-posedness complicates the problem even further, as more data samples are required to obtain a positive-definite covariance estimate, with at least \( \frac{1}{2} (N^2 + N) \) independent and identically distributed (i.i.d.) observations needed.

For enhanced intuition, an approach referred to as common PCA (Flury 1988) aims to extract the eigenvectors that span a space that is identical across the interest rate curves of multiple countries (common latent factors). However, this method relies on the simultaneous diagonalization of multiple covariance matrixes, which is in general both: 1) not tractable for more than two matrixes, and 2) unable to model the co-variation between assets across countries (Juneja 2012). To this end, inter-battery factor analysis (Tucker 1958) was employed to capture all common factors across domestic term structures (Perignon et al. 2007). The method is computationally prohibitive, and implicitly assumes that idiosyncratic co-variations can occur only domestically, a restrictive and unrealistic assumption.

All in all, the main limitation underpinning PCA-based techniques lies in their reliance on standard matrix methods developed for multivariate analysis, which are agnostic to the intrinsic multi-curve lattice data structure inherent to real-world global fixed income returns, as shown in Exhibit 1. Such a flattened view of the data, and the limitations of the use of intrinsically pair-wise (scalar-to-scalar) based linear matrix algebra in multivariate analysis, make current strategies by and large inadequate and ineffective.

In order to address this void and account for the inter-related maturity vs. country nature of fixed income returns, we propose a multilinear algebraic approach, with the aim of matching the underlying data structure. We show that this allows for a rigorous model of global fixed income returns, spanning multiple maturities and countries, as in this way the rigid lattice structures where such data reside are accounted for in a natural and compact way. Indeed, multilinear algebra is a backbone of multiway tensor analysis that has found its place in a number of real-world applications across science and engineering (Kolda and Bader 2009; Cichocki et al. 2015, 2017a,b; Siridopoulos et al. 2017), as these techniques allow us to develop appropriate models for capturing the structured linear couplings and interactions between data fibers (multidimensional generalization of matrix rows and columns), which are intrinsic to lattice-structured data. In addition, by virtue of the proposed multilinear model, the covariance of returns between two assets becomes linearly separable, that is, it factorizes into a product of a maturity-to-maturity covariance and a country-to-country covariance, thus giving the analysis enhanced physical meaning. This is in stark contrast to the standard linear algebraic approach, which only can model each pair-wise asset-to-asset covariance. In other words, the proposed multilinear approach allows us to decompose

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**EXHIBIT 1**

The Lattice Data Structure Inherent to the Interest Swap Rates\(^6\) for Eight Developed Countries: Switzerland, Euro Area, United Kingdom, Japan, Australia, New Zealand, Canada, and the United States, with Maturities of \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 25, 30\} \) Years, Observed on July 22, 2017

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**SOURCE:** Bloomberg.
the covariance matrix of global fixed income returns into a multilinear (Kronecker) product of the two matrixes associated with the independent covariance domains:

1. **Maturity-domain** covariance exhibited within all countries and
2. **Country-domain** covariance displayed at all maturities.

Operating within each domain in parallel makes it possible to:

- Devise novel rigorous and tractable global portfolio management strategies, which are specifically tailored to each covariance domain
- Parametrize the global structure of the covariance of fixed-income returns with fewer decision parameters, compared to standard linear models
- Provide a rigorous and parsimonious alternative to the classical unrestricted estimate, which is unstable or even unavailable if the number of assets considered is large compared to the sample size.

Our empirical analysis confirms the utility of the maturity-domain and country-domain covariances so derived, underlying the interest rate curves of eight developed economies, in providing a compact and physically meaningful insight into the global macroeconomic environment.

**PRELIMINARIES OF MULTILINEAR ALGEBRA**

A class of multi-dimensional arrays referred to as tensors, which are endowed with multilinear properties, is introduced following the notation in Kolda and Bader (2009), whereby scalars are denoted by a lightface font, e.g., \( x \); vectors by a lowercase boldface font, e.g., \( x \); matrixes by an uppercase boldface font, e.g., \( X \); and tensors by a boldface calligraphic font, e.g., \( \mathcal{X} \).

The order of a tensor is defined as the number of its dimensions, also referred to as modes, i.e., the tensor \( \mathcal{X} \in \mathbb{R}^{l_1 \times l_2 \times \cdots \times l_N} \) has \( N \) modes and \( K = \prod_{n=1}^{N} l_n \) entries in total. Tensors can be reshaped into lower-dimensional representations (unfoldings) that can be manipulated using standard linear algebra. The vector unfolding, also known as vectorization, is denoted by

\[
  x = \text{vec}(\mathcal{X}) \in \mathbb{R}^K
\]

with its elements given by

\[
  [x]_{\overline{1} \overline{2} \cdots \overline{l_N}} = \left[ \mathcal{X} \right]_{l_1, l_2, \ldots, l_N}
\]

Note that we use overlined subscripts to denote the linear indexing (or Little-Endian) convention (Kolda and Bader 2009), whereby

\[
  \overline{l_1 l_2 \cdots l_N} = 1 + \sum_{n=1}^{N} (i_n - 1) l_n
\]

The mode-\( n \) unfolding (matricization), denoted by \( X_{(n)} \in \mathbb{R}^{l_n \times K} \), is obtained by reshaping a tensor into a matrix in the form

\[
  X_{(n)} = \begin{bmatrix}
    f_1^{(n)} & f_2^{(n)} & \cdots & f_K^{(n)}
  \end{bmatrix}
\]
where the column vector, $f^{(n)}_i \in \mathbb{R}^K$, is the $i$-th mode-$n$ fiber (a multi-dimensional generalization of matrix rows and columns), with elements defined as

$$f^{(n)}_i = [\mathcal{X}]_{i_1 \cdots i_{n-1} i_n \cdots i_K}$$

with $i$ as the linear index $i_1 \cdots i_{n-1} i_{n+1} \cdots i_K$, that is

$$i = 1 + \sum_{k=1}^{N} (i_k - 1) k$$

(6)

The operation of mode-$n$ unfolding also can be viewed as a rearrangement of the mode-$n$ fibers into column vectors of the matrix, $X(n)$, as illustrated in Exhibit 2. Notice that the considered order-3 tensor, $\mathcal{X}$, has alternative representations in terms of mode-1 (left panel), mode-2 (middle panel), and mode-3 (right panel) fibers, that is, its columns, rows, and tubes.

We next consider the second-order moments of a tensor-valued random variable, which, without loss of generality, is assumed to have a zero-mean. The variance of a tensor-valued random variable, $\mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$, is defined as its expected squared Frobenius norm, that is

$$\text{var}(\mathcal{X}) = E(\| \mathcal{X} \|^2_F) \in \mathbb{R}$$

(7)

Using the fiber-based mode-$n$ unfolding representation in Equation 4, it is possible to define the mode-$n$ covariance matrix as follows

$$\text{cov}(X(n)) = E(X(n)X(n)^T) \in \mathbb{R}^{K \times K}$$

(8)

Intuitively, the mode-$n$ covariance in Equation 8 can be viewed as the expected covariance between mode-$n$ fibers, owing to the total expectation theorem (Weiss et al. 2005), since

$$E_{(n)}(X(n)X(n)^T) = \sum_{i=1}^{K} E_{(n)}(f^{(n)}_i f^{(n)T}_i) = \frac{K}{l_n} E_i[\text{cov}(f^{(n)})] = \frac{K}{l_n} \text{cov}(f^{(n)})$$

(9)

where $E_{(i)}$ denotes the expectation over the indexes $i$. Intuitively, the $i$-th mode-$n$ fiber, $f^{(n)}_i$, can be viewed as the $i$-th realization of a stable generic i.i.d. random variable, $f^{(n)}$. Note that we assume ergodicity over the space associated with the indexes, $i$, which allows us to introduce an expectation operator through summation.

It is important to highlight that, although we consider order-2 tensors (matrixes), the multilinear matrix model proposed in this work largely differs from existing standard linear matrix models, as shown next.

**A MULTILINEAR MODEL FOR GLOBAL FIXED INCOME RETURNS**

In standard multivariate data analysis, multiple measurements are collected at a given trial, experiment, or time instant, to form a vector-valued data sample, $x \in \mathbb{R}^K$. 
A prior assumption typically adopted in statistical modeling is that the observables are distributed according to a normal distribution, \( x \sim \mathcal{N}(\mu, \Sigma) \), which implies that the mean vector, \( \mu \in \mathbb{R}^n \), and covariance matrix, \( \Sigma \in \mathbb{R}^{n \times n} \), are in general of an “overall” nature and thus unstructured. However, if the entries in \( x \) lie on a regular lattice in an \( N \)-dimensional coordinate system, then it is desirable, and often even necessary, to assume that the mean and covariance exhibit a more structured form that arises from relevant physical considerations (Hoff 2011; Gupta and Nagar 2000; Scalzo et al. 2020). This is precisely the case for global fixed income returns that, as shown in Exhibit 1, reside in a two-dimensional coordinate system (maturity \( \times \) country), as demonstrated next.

### Multilinear Model of Global Fixed Income Returns

Consider a zero-mean random variable, \( x(m, c) \in \mathbb{R} \), which represents the return of a fixed income asset with a maturity, \( m \), in a country, \( c \), at a time instant \( t \). In other words, the fixed income return, \( x: \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R} \), is a bilinear map that maps a maturity, \( m \), and a country, \( c \), to a realization of a random variable \( x(m, c) \) (a return). It can be interpreted as a random scalar field in a two-dimensional coordinate system spanned by the maturity and country axes.

For generality, we assume that the returns are distributed according to the coordinate-dependent distribution

\[
x(m, c) \sim \mathcal{N}(0, \sigma^2(m, c)) \tag{10}
\]

so that the covariance operator, \( \sigma: \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R} \), of the return field is defined as

\[
\sigma(m, c, m', c') = \text{cov}(x(m, c), x(m', c')) \tag{11}
\]

with \( \sigma(m, c, m', c') = \sigma^2(m, c) \).

Such a random field is said to exhibit a linearly separable covariance operator if it can be factorized into a product of covariance operators of constituent random variables, that is, each associated with an axis (country or maturity) of the underlying coordinate system. Then, a linearly separable covariance operator of the global returns in Equation 10 takes the form

\[
\sigma(m, c, m, c) = \sigma^{(m)}(m, m')\sigma^{(c)}(c, c') = \sigma^{(m)}\sigma^{(c)}, \quad \forall i, j, k, l \tag{12}
\]

where \( \sigma^{(m)} \) is the covariance between asset returns with maturities \( m_i \) and \( m_k \) (which is independent of the countries), while \( \sigma^{(c)} \) is the covariance between asset returns in countries \( c_j \) and \( c_l \) (which is independent of the maturities).

### Reshaping of the Global Return Data

When jointly considering the returns of assets at \( l_m \) maturities within \( l_c \) countries at a time instant \( t \), the returns can be shaped up to form an order-2 tensor (matrix-valued) random variable, \( X_t \in \mathbb{R}^{l_m \times l_c} \), with its \((i, j)\)-th entry defined as

\[
[X_{ij}]_t = x_i(m, c_j), \quad i = 1, \ldots, l_m, \quad j = 1, \ldots, l_c. \tag{13}
\]

Therefore, each realization of the matrix, \( X_t \), contains \( l_ml_c \) entries in total; an illustration of this matrix in a tensor-valued formalism at a time-instant, \( t \), is given in Exhibit 3.
We next demonstrate that the vector and matrix unfoldings of the considered tensor in Exhibit 1 admit a physically meaningful interpretation. To this end, observe that the mode-1 and mode-2 unfoldings of \( X_t \), that is, \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \), take the form

\[
\mathcal{X}_1 = \begin{bmatrix} x(m_1, c_1) \\
\vdots \\
x(m_{Ic}, c_{Ic}) \end{bmatrix} \quad \text{and} \quad \mathcal{X}_2 = \begin{bmatrix} x(m_1, c_1) \\
\vdots \\
x(m_{Ic}, c_{Ic}) \end{bmatrix}
\]

(see Exhibits 2 and 3)

with the \( j \)-th mode-1 fiber, \( f_j^{(1)} \), containing the returns of all maturities associated with the \( j \)-th country (the returns of an entire domestic curve). Similarly, the \( i \)-th mode-2 fiber, \( f_i^{(2)} \), containing the returns of all countries associated with the \( i \)-th maturity; see Exhibit 3 for an illustration. Finally, observe from Exhibit 3 that the vector unfolding of global returns, \( X_t \), can be expressed in terms of mode-1 fibers (domestic yield curves) as shown in Exhibit 4.

Second-order Moments of Global Returns through Multilinear Algebra

At the core of our approach is the assumption that if data exhibit the lattice structure in Exhibit 1, then multilinear algebra can be used to naturally decompose the covariance matrix of global fixed income returns, \( \mathbf{E} \), into its maturity-domain and country-domain components. To justify this assumption, observe that, from Exhibit 1 and the condition in Equation 12, if the entries in \( X_t \) are sampled from a linearly separable field, then the vectorized form of tensor-valued random variable, that is \( x_t = \text{vec}(X_t) \), exhibits a Kronecker separable covariance structure of the form (Hoff 2011; Scalzo et al. 2020).

**EXHIBIT 3**

Illustration of an Order-2 Tensor (Matrix) Sample, Represented in Terms of: (i) the Individual Returns, \( x(m_i, c_j) \), as in Matrixes (Left Panel); (ii) Mode-1 Fibers of a Tensor (Matrix Columns), \( f_j^{(1)} \) for \( j = 1, \ldots, Ic \), (Middle Panel); and (iii) Mode-2 Fibers of a Tensor (Matrix Rows), \( f_i^{(2)} \) for \( i = 1, \ldots, Im \), (Right Panel).

**EXHIBIT 4**

Illustration of the Vectorisation of an Order-2 Tensor (Matrix) Sample, Represented in Terms of Mode-1 Fibers (Domestic Yield Curves), \( f_j^{(1)} \) for \( j = 1, \ldots, Ic \), with the \( j \)-th mode-1 fiber, \( f_j^{(1)} \), containing the returns of all maturities associated with the \( j \)-th country (the returns of an entire domestic curve). Similarly, the \( i \)-th mode-2 fiber, \( f_i^{(2)} \), containing the returns of all countries associated with the \( i \)-th maturity; see Exhibit 3 for an illustration. Finally, observe from Exhibit 3 that the vector unfolding of global returns, \( x_t \), can be expressed in terms of mode-1 fibers (domestic yield curves) as shown in Exhibit 4.
with $\Sigma^{(m)} \in \mathbb{R}^{l_1 \times l_2}$ as the maturity-domain covariance matrix and $\Sigma^{(c)} \in \mathbb{R}^{l_3 \times l_4}$ the country-domain covariance matrix. From Equation 12 we also have

$$[\Sigma^{(m)}]_{ij} = \sigma^{(m)}_{ij}$$  \hspace{1cm} (18)

and therefore the Kronecker separable structure in Equation 17 can be expanded as

$$E\{x, x^T\} = \sum_{ii} \Sigma^{(c)} \sum_{jj} \Sigma^{(m)} = \text{tr}(\Sigma^{(c)}) \text{tr}(\Sigma^{(m)})$$  \hspace{1cm} (20)

The linear separability condition in Equation 12 also implies the following relationships concerning the second-order moment of every mode-$n$ unfolding

$$E\{X, X^T\} = \text{tr}(\Sigma^{(c)}) \Sigma^{(m)}$$  \hspace{1cm} (21)

$$E\{X^T, X\} = \text{tr}(\Sigma^{(m)}) \Sigma^{(c)}$$  \hspace{1cm} (22)

This can be proved by inspecting the entries of the second-order moments of the matrix unfoldings in Exhibit 3 and Equations 14–15. For example, for the mode-1 unfolding, we have

$$E\{XX^T\}_{jk} = \sum_{i=1}^{l_1} \sum_{j=1}^{l_2} \sigma^{(m)}_{ij} \sigma^{(c)}_{jk} = \text{cov}(x(m, c), x(m', c')) = \sum_{i=1}^{l_1} \sigma^{(m)}_{ij} \sigma^{(c)}_{jk} = [\Sigma^{(m)}]_{jk} \text{tr}(\Sigma^{(c)})$$  \hspace{1cm} (23)

Similarly, we arrive at the result in Equation 22 for the mode-2 unfolding, from

$$E\{X^T X\}_{jk} = \sum_{i=1}^{l_3} \sum_{j=1}^{l_4} \sigma^{(m)}_{ij} \sigma^{(c)}_{jk} = \text{cov}(x(m', c), x(m, c)) = \sum_{i=1}^{l_3} \sigma^{(m)}_{ij} \sigma^{(c)}_{jk} = [\Sigma^{(m)}]_{jk} \text{tr}(\Sigma^{(c)})$$  \hspace{1cm} (24)

Therefore, $E\{XX^T\}_{jk}$ is proportional to the covariance between asset returns with maturities $m_j$ and $m_k$ for a given country, averaged over all countries, while $E\{X^T X\}_{jk}$ is proportional to the covariance between asset returns in countries $c_j$ and $c_k$ at a given maturity, averaged over all maturities.

Remark 1. The concept underpinning the proposed multilinear model is that the covariance matrix, $E\{xx^T\}$, of the global returns admits a characterization in terms of its fiber-to-fiber (multilinear) covariance parameters. This contrasts the entry-to-entry (linear) covariance parameters implied by the multivariate normal distribution and assumed in standard linear algebra based matrix models.

In addition, the Kronecker separability condition in Equation 12, which is inherent to multilinear algebra, provides a stable and parsimonious alternative to an unconstrained standard estimate of $E\{xx^T\}$, which is unstable or even unavailable if the number of variables (assets) are large compared to the number of samples. For example, consider the unconstrained covariance matrix, $\Sigma \in \mathbb{R}^{l_1 \times l_2}$,
calculated based on a standard multivariate treatment of the global returns data, which (owing to its symmetry) has \( \frac{1}{2}(l_m^2 + l_c^2) \) distinct entries. In contrast, its Kronecker separable counterpart, \( \Sigma^{(c)} \otimes \Sigma^{(m)} \), reduces to only \( \frac{1}{2}(l_m^2 + l_c^2 + l_m + l_c) \) distinct parameters. To fully appreciate the scale of dimensionality reduction achieved by the proposed multilinear model, consider fixed income returns for \( I_m = 15 \) maturities and \( I_c = 8 \) countries, that is, with \( l_m l_c = 120 \) returns in total at any time instant. Then, the standard multivariate covariance matrix will have \( \frac{1}{2}(120^2 + 120) = 7260 \) distinct parameters, whereas the Kronecker separable multilinear counterpart reduces the model to \( \frac{1}{2}(15^2 + 15 + 8^2 + 8) = 156 \) parameters, a reduction by a factor of 97.9%. For more insight into the practical utility of the so achieved parameter reduction, see Exhibit 5.

### Implied Domestic and Cross-country Covariance

Now that we have illustrated the benefits of Kronecker separability conditions, we proceed to investigate the domestic and cross-country fixed income return interactions implied by the proposed multilinear model. To this end, observe that the covariance matrix of global returns, \( \Sigma = \mathbb{E}(\mathbf{x}_t \mathbf{x}_t') \), takes the form of a block matrix containing both domestic and cross-country covariance matrices, that is

\[
\Sigma = \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1l_c} \\
\Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2l_c} \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_{l_c,1} & \Sigma_{l_c,2} & \cdots & \Sigma_{l_c l_c}
\end{bmatrix}
\]

(25)

where \( \Sigma_{ij} \in \mathbb{R}^{l_m \times l_m} \) is the domestic interest rate curve covariance matrix associated with the \( i \)-th country, while \( \Sigma_{ij} \in \mathbb{R}^{l_m \times l_m} \) is the cross-country interest rate curve covariance matrix between the \( i \)-th and \( j \)-th countries. From the Kronecker separability condition in Equation 20, the multilinear model asserts that each domestic and cross-country covariance matrix of returns takes the form

\[
\Sigma_{ij} = \sigma^{(c)}_{ij} \Sigma^{(m)}
\]

(26)

which implies that all countries share the same maturity-domain covariance component, \( \Sigma^{(m)} \). In other words, all countries exhibit the same level, slope, and curvature factors, as demonstrated empirically in the Global Analysis Section below.
A Statistically Identifiable Formulation

It is important to highlight that, although the introduced parametrization of the covariance decomposition in Equation 17 is physically meaningful, it is statistically non-identifiable, that is, different values of $\Sigma^{(m)}$ and $\Sigma^{(c)}$ may generate the same Kronecker product, $\Sigma^{(c)} \otimes \Sigma^{(m)}$ (Dutilleul 1999). On the other hand, the identifiability condition is absolutely necessary for a parameter estimator to be statistically consistent (Newey and McFadden 1994), as it asserts that the associated log-likelihood function has a unique global maximum (Huzurbazar 1947).

To overcome this issue, Scalzo et al. (2020) introduced a statistically identifiable formulation of the Kronecker separable covariance parameters, to yield statistically consistent estimators. For our case, this yields the following statistically identifiable re-parametrization of the covariance parameters

$$\Sigma^{(c)} \otimes \Sigma^{(m)} = \sigma^2 (\Theta^{(c)} \otimes \Theta^{(m)}) \quad (27)$$

where $\sigma^2 \in \mathbb{R}$ denotes the variance of the tensor-valued random variable of global returns, defined as

$$\sigma^2 = \text{tr} (\Sigma^{(c)} \otimes \Sigma^{(m)}) = \text{tr} (\Sigma^{(m)}) \text{tr} (\Sigma^{(c)}) \quad (28)$$

while $\Theta^{(m)} \in \mathbb{R}^{I_m \times I_m}$ and $\Theta^{(c)} \in \mathbb{R}^{L \times L}$ denote respectively the maturity-domain and country-domain covariance density matrices, which, being density measures, are constrained to have unit-trace, that is

$$\text{tr} (\Theta^{(m)}) = \text{tr} (\Theta^{(c)}) = 1 \quad (29)$$

Now, under the identifiable parametrization in Equation 27, the second-order moments of the tensor-valued random variable become

$$E \{ \| X \|_F^2 \} = \sigma^2 \quad (30)$$

$$E \{ X^T X \} = \sigma^2 (\Theta^{(c)} \otimes \Theta^{(m)}) \quad (31)$$

with

$$E \{ X^T X \} = \sigma^2 \Theta^{(m)} \quad (32)$$

$$E \{ X^T X \} = \sigma^2 \Theta^{(c)} \quad (33)$$

Intuitively, $\sigma^2$ is the expected Frobenius norm of the tensor of global fixed income returns, while the covariance density matrices, $\Theta^{(m)}$ and $\Theta^{(c)}$, designate respectively the proportion of the total variance, $\sigma^2$, associated with each mode-1 fiber (domestic curve) and mode-2 fiber (cross-country constant-maturity assets).

Estimation Procedure

Using the proposed identifiable framework, the estimate of the “tensor variance” parameter, $\sigma^2$, over $T$ time instants is given by

$$\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^{T} \| X_t \|_F^2 \quad (34)$$
while the maturity-domain and country-domain covariance density matrixes are obtained as

$$\Theta^{(m)} = \frac{1}{\sigma^2} \sum_{t=1}^{T} X_t X_t^T$$  \hspace{1cm} (35)

$$\Theta^{(c)} = \frac{1}{\sigma^2} \sum_{t=1}^{T} X_t^c X_t^c$$  \hspace{1cm} (36)

Observe that these are sample estimators of a method of moments type, which are also statistically consistent (Scalzo et al. 2020).

GLOBAL FIXED INCOME FACTOR ANALYSIS

The maturity-domain and country-domain covariance density matrixes established in the previous section now can be employed to yield orthogonal basis functions that are obtained through their corresponding eigenvalue decompositions

$$\Theta^{(m)} = U^{(m)} \Lambda^{(m)} U^{(m)T}$$  \hspace{1cm} (37)

$$\Theta^{(c)} = U^{(c)} \Lambda^{(c)} U^{(c)T}$$  \hspace{1cm} (38)

Notice that, owing to the properties of the Kronecker product (Magnus and Neudecker 1985), while the principal component analysis of the global returns covariance matrix still yields the standard eigendecomposition

$$E \{X_t X_t^T\} = U \Lambda U^T$$  \hspace{1cm} (39)

the matrices of eigenvectors, $U$, and eigenvalues, $\Lambda$, now become Kronecker separable, that is

$$U = (U^{(c)} \otimes U^{(m)})$$  \hspace{1cm} (40)

$$\Lambda = \sigma^2 (\Lambda^{(c)} \otimes \Lambda^{(m)})$$  \hspace{1cm} (41)

Remark 2. We refer to the decomposition in Equations 39–41 as multilinear PCA. This result is intrinsically linked to the well-known multilinear singular value decomposition (Tucker 1966 and De Lathauwer et al. 2000).

The maturity-domain eigenvector matrix, $U^{(m)} \in \mathbb{R}^{I \times m}$, contains column vectors, $u^{(m)}_i \in \mathbb{R}^{I}$, that describe the orthogonal directions of the maturity-domain covariance. These vectors include the well-known level, slope, and curvature factors. Similarly, $U^{(c)} \in \mathbb{R}^{I \times c}$ contains column vectors, $u^{(c)}_i \in \mathbb{R}^{I}$, which describe orthogonal directions of the country-domain covariance. These also have an economically meaningful interpretation, as shown in the sequel.

Note that for both factors, $U^{(m)}$ and $U^{(c)}$, the eigenvectors are orthogonal, that is, $U^{(m)T} U^{(m)} = I$ and $U^{(c)T} U^{(c)} = I$. Similarly, the diagonal matrix, $\Lambda^{(m)} \in \mathbb{R}^{I \times I}$, contains the eigenvalues, $\lambda^{(m)}_i$, which describe the fraction of the total variance, $\sigma^2$, explained by factor $u^{(m)}_i$, while $\Lambda^{(c)} \in \mathbb{R}^{I \times I}$, contains the eigenvalues, $\lambda^{(c)}_i$, on its diagonal, which describe the fraction of the total variance, $\sigma^2$, explained by factor $u^{(c)}_i$. As such, the eigenvalues of both $\Lambda^{(m)}$ and $\Lambda^{(c)}$ sum up to unity, i.e., $\text{tr} (\Lambda^{(m)}) = \text{tr} (\Lambda^{(c)}) = 1$. 
Owing to the Kronecker structure in Equation 39–41, upon inspecting the \( i \)-th eigenvector, \( \mathbf{u}_i \in \mathbb{R}^{l_c l_r} \), of the matrix \( \mathbf{U} \in \mathbb{R}^{l_c l_r \times l_c l_r} \) in Equation 39, observe that it also exhibits the following Kronecker structured form

\[
\mathbf{u}_i = (\mathbf{u}_i^{(c)} \otimes \mathbf{u}_i^{(m)}) = \begin{bmatrix}
\mathbf{u}_i^{(c)} \mathbf{u}_i^{(m)} \\
\mathbf{u}_i^{(c)} \mathbf{u}_i^{(m)} \\
\vdots \\
\mathbf{u}_i^{(c)} \mathbf{u}_i^{(m)}
\end{bmatrix}
\]  

(42)

with \( i = 1 + (j - 1)l_m \), and \( u_i^{(c)} \) as the \( l \)-th element of the vector \( \mathbf{u}_i^{(c)} \). Consequently, the eigenvalue, \( \lambda_i \), is also separable and is given by

\[
\lambda_i = \lambda_i^{(c)} \lambda_i^{(m)}
\]  

(43)

From the structure of the eigenvector in Equation 42, we can assert that the eigenvector \( \mathbf{u}_i \) represents a concatenation of the maturity-domain components, \( \mathbf{u}_i^{(m)} \), which are the same for each country, each weighted by their associated country-domain scaling factor, whereby the \( l \)-th country is scaled by \( u_i^{(c)} \in \mathbb{R} \). This follows from the result in Equation 26, that is, the \((i, j)\)-th cross-country covariance matrix admits the spectral expansion in the form

\[
\Sigma = \sum_{k=1}^{l_m} \sum_{l=1}^{l_c} \lambda_k \lambda_l \mathbf{u}_k^{(c)} \mathbf{u}_l^{(c)} \mathbf{u}_i^{(m)} \mathbf{u}_i^{(m) T}
\]  

(44)

GLOBAL PORTFOLIO MANAGEMENT AND HEDGING

The application of principal components for a domestic interest rate curve analysis has been widely studied and implemented by the investment community for both the evaluation of market risk and construction of hedged portfolios (SalomonSmithBarney 2000; Credit Suisse 2012; Standard Chartered 2013; TD Securities 2015). However, existing approaches do not account for the sources of risk common to multiple interest rate curves—a critical concept required for an appropriate assessment of the potential for international diversification within a global fixed income portfolio.

Consider a global fixed income portfolio, consisting of assets with \( l_m \) distinct maturities within \( l_c \) distinct countries, that is, \( l_m l_c \) assets in total. As the measure of portfolio risk, we consider the portfolio variance, \( \sigma_p^2 \), which is a function of the global covariance matrix of returns, \( \Sigma = \mathbb{E} \{ \mathbf{x}_i \mathbf{x}_j^T \} \in \mathbb{R}^{l_c l_r \times l_c l_r} \). Given a vector of portfolio weights, \( \mathbf{w} \in \mathbb{R}^{l_c l_r} \), the portfolio variance can be expressed as

\[
\sigma_p^2 = \mathbf{w}^T \Sigma \mathbf{w}
\]  

(45)

We have demonstrated in the previous section that, when considering an international basket of fixed income assets, the covariance, \( \Sigma = \sigma^2 (\Theta^{(c)} \otimes \Theta^{(m)}) \), exhibits a Kronecker separable structure. In the same spirit, since Exhibit 1 suggests that our portfolio vector also exhibits a Kronecker separable structure, we can formalize this as

\[
\mathbf{w} = \mathbf{w}^{(c)} \otimes \mathbf{w}^{(m)}
\]  

(46)
where \( w^{(m)} \in \mathbb{R}^{I_m} \) and \( w^{(c)} \in \mathbb{R}^{I_c} \) are respectively the maturity-domain and country-domain portfolio weights. As shown in Exhibit 5, this yields parsimonious and compact solutions for global portfolio management schemes. The corresponding reduction in parameters required to optimize the portfolio—from the standard \( I_{mI_c} \) to \( (I_m + I_c) \) in the proposed scheme—is only made possible through the multilinear decomposition of the global returns covariance into the maturity-domain and country-domain covariances, as shown in Equation 31.

**Minimum Variance Portfolio through Multilinear Analysis**

Consider the standard capital-constrained portfolio setup, which attains the minimum variance through the widely studied optimization problem, given by (Markowitz 1952)

\[
\min_w w^T \Sigma w, \quad \text{s.t. } w^T 1 = 1
\]  

(47)

with \( 1 \) as a vector of ones. The solution is given by the well-known minimum variance portfolio

\[
w_{opt} = \frac{\Sigma^{-1} 1}{1^T \Sigma^{-1} 1}.
\]  

(48)

From Equation 31, for the considered Kronecker separable case, the optimal multilinear portfolio allocation reduces to

\[
w_{opt} = \frac{(\Theta^{(c)} \otimes \Theta^{(m)})^{-1} 1}{1^T (\Theta^{(c)} \otimes \Theta^{(m)})^{-1} 1} = \left( \frac{(\Theta^{(c)})^{-1} 1}{1^T (\Theta^{(c)})^{-1} 1} \right) \otimes \left( \frac{(\Theta^{(m)})^{-1} 1}{1^T (\Theta^{(m)})^{-1} 1} \right) = w^{(c)}_{opt} \otimes w^{(m)}_{opt}
\]  

(49)

**Remark 3.** The result in Equation 49 asserts that portfolio optimization can be separated into parallel sub-problems that operate within their respective maturity and country domains. This is equivalent to solving for \( w^{(m)} \) and \( w^{(c)} \) independently through the following individual minimizations

\[
\min_{w^{(m)}} w^{(m)T} \Theta^{(m)} w^{(m)}, \quad \text{s.t. } w^{(m)T} 1 = 1
\]

and

\[
\min_{w^{(c)}} w^{(c)T} \Theta^{(c)} w^{(c)}, \quad \text{s.t. } w^{(c)T} 1 = 1
\]

The asset allocation associated with a fixed income asset within the \( j \)-th country and with the \( k \)-th maturity now can be expressed in an intuitive and physically meaningful way as

\[
[w]_i = [w^{(c)}]_j [w^{(m)}]_k
\]  

(50)

with \( i = k + (j - 1)I_m \). Also notice that the proposed model significantly alleviates the computational burden associated with the minimum-variance portfolio solution, since the original inversion of the matrix \( \Sigma \) with dimensions \( (I_m \times I_m) \) is replaced with the inversion of two smaller matrixes, \( \Theta^{(c)} \) and \( \Theta^{(m)} \), with the respective dimensions \( (I_c \times I_c) \) and \( (I_m \times I_m) \).
Hedging

The task of hedging fixed income securities remains one of the most challenging problems faced by financial institutions. In the context of hedging, the $i$-th principal component of the global returns covariance matrix, $\mathbf{u}_i \in \mathbb{R}^{m \times 1}$, can be viewed as a portfolio allocation vector associated to a latent market risk factor. Therefore, the aim of hedging is to minimize the risk exposure of a portfolio allocation, $\mathbf{w}$, with respect to an $i$-th principal component, $\mathbf{u}_i$, that is given by their inner product, $\mathbf{u}_i \mathbf{w}$. As a consequence, a hedged portfolio is attained through the condition of orthogonality, that is, $\mathbf{u}_i \mathbf{w} = 0$.

Based on Equation 42, within the considered multi-country setup every principal component of the global covariance matrix decomposes into:

$$
\mathbf{u}_i \mathbf{w} = (\mathbf{u}_i^{(c)} \mathbf{w}^{(c)})(\mathbf{u}_i^{(m)} \mathbf{w}^{(m)})
$$

(51)

**Remark 4.** The multilinear decomposition of the portfolio risk exposure in Equation 51 asserts that the desired condition of orthogonality for hedging, $\mathbf{u}_i \mathbf{w} = 0$, can be attained by either achieving orthogonality within the maturity domain or within the country domain, and thus it is not necessary to achieve orthogonality within both domains simultaneously. This is because, if either $\mathbf{u}_i^{(c)} \mathbf{w}^{(c)} = 0$ or $\mathbf{u}_i^{(m)} \mathbf{w}^{(m)} = 0$, the global exposure in Equation 51 vanishes.

We next illustrate the advantages of this multilinear risk exposure result in real-world applications.

**Hedging long-term bonds.** Consider hedging a long-only portfolio of international long-term fixed income assets, all of which have maturity $m$ (e.g., 30 years), using an international portfolio of short-term assets. The hedged portfolio therefore must satisfy the following well-known conditions within the maturity domain:

$$
\delta_i^T \mathbf{w}^{(m)} = 1
$$

(52)

$$
\text{self-financing} \quad \mathbf{1}^T \mathbf{w}^{(m)} = 0
$$

(53)

orthogonality w.r.t. leading $n$ maturity-domain PCs $\mathbf{u}_j^{(m)} \mathbf{w}^{(m)} = 0$, $j = 1, \ldots, n$

(54)

where $\delta_i \in \mathbb{R}^n$ is a vector of zeros except for the $i$-th element which equal to 1. Intuitively, the first condition reflects the long-only position in assets with the $i$-th maturity; the second condition constrains the strategy to be self-financing; and the last condition enforces orthogonality with respect to the leading $n$ maturity-domain principal components.

**Hedging domestic bonds.** Conversely, consider hedging a domestic long-only portfolio within country $c$, using an international portfolio. The hedged portfolio must satisfy the following constraints within the country domain:

$$
\delta_i^T \mathbf{w}^{(c)} = 1
$$

(55)

**EXHIBIT 6**

Explanatory Power [%] of PC1 (Level), PC2 (Slope) and PC3 (Curvature) for the Eight Economies Considered

<table>
<thead>
<tr>
<th>Economy</th>
<th>Level</th>
<th>Slope</th>
<th>Curvature</th>
</tr>
</thead>
<tbody>
<tr>
<td>SF</td>
<td>87.88</td>
<td>10.02</td>
<td>1.16</td>
</tr>
<tr>
<td>EU</td>
<td>94.15</td>
<td>4.78</td>
<td>0.66</td>
</tr>
<tr>
<td>GB</td>
<td>95.29</td>
<td>3.83</td>
<td>0.56</td>
</tr>
<tr>
<td>JP</td>
<td>82.04</td>
<td>14.10</td>
<td>2.28</td>
</tr>
<tr>
<td>AU</td>
<td>92.84</td>
<td>4.94</td>
<td>0.95</td>
</tr>
<tr>
<td>NZ</td>
<td>92.30</td>
<td>5.76</td>
<td>0.87</td>
</tr>
<tr>
<td>CA</td>
<td>93.14</td>
<td>5.74</td>
<td>0.66</td>
</tr>
<tr>
<td>US</td>
<td>95.30</td>
<td>4.06</td>
<td>0.47</td>
</tr>
</tbody>
</table>
self-financing \( 1^T w^{(c)} = 0 \)  

(56)

orthogonality w.r.t. leading \( n \) country-domain PCs \( u_j^{(c)T} w^{(c)} = 0, \ j = 1, \ldots, n \)  

(57)

where \( \delta_i \in \mathbb{R}^k \) is a vector of zeros except for the \( i \)-th element which equal to 1. The first condition above reflects the long-only position in the \( i \)-th country; the second condition constrains the strategy to be self-financing; and the last condition enforces orthogonality with respect to the leading \( n \) country-domain principal components.

The above portfolio hedging problems reduce to solving the following linear systems

\[
A^{(m)} w^{(m)} = \begin{bmatrix}
\delta_i^T \\
1^T w^{(m)} \\
\vdots \\
u_i^{(m)T}
\end{bmatrix} \begin{bmatrix}
w^{(m)} \\
1 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} = b^{(m)}  
\]

(58)

\[
A^{(c)} w^{(c)} = \begin{bmatrix}
\delta_i^T \\
1^T w^{(c)} \\
\vdots \\
u_i^{(c)T}
\end{bmatrix} \begin{bmatrix}
w^{(c)} \\
1 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} = b^{(c)}  
\]

(59)

Based on Equations 58 and 59, the optimal maturity-domain and country-domain weights are therefore given by \( w^{(m)} = A^{(m)} b^{(m)} \) and \( w^{(c)} = A^{(c)} b^{(c)} \), where \((\cdot)^+\) denotes the Moore-Penrose inverse operator.

**EMPIRICAL ANALYSIS**

We next provide a validation of the proposed multilinear factor model through an empirical analysis of the global term structure of the international interest rate swaps (IRS) market. The data comprise weekly IRS rate curves\(^1\), ranging from January 1, 2015, to July 1, 2019, for eight developed economies: Switzerland, Euro Area, United Kingdom, Japan, Australia, New Zealand, Canada, and the United States. Each domestic IRS curve consisted of swaps with maturities \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 25, 30\} \) years. Therefore, every week we observed \( I_m = 15 \) IRS returns for each of the \( I_c = 8 \) economies, resulting in \( I_m I_c = 120 \) weekly observations in total. Exhibit 7 displays the historical IRS rates employed in the analysis. Observe the similar collective behavior of the historical data that confirms the importance of global factors in driving the co-movement of fixed income securities across advanced economies.

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\(^1\)The swap rate is the fixed interest rate that the receiver of the IRS demands in exchange for the uncertainty of having to pay the short-term floating LIBOR rate over time.
EXHIBIT 7
Weekly Swap Ratesa for the Eight Economies Considered, with the Maturities of (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 25, 30) Years (Respectively Color-Coded from Blue to Red) for the Period 2015-01-01 to 2019-07-01 (Left Panel) and Their Corresponding PC1 (Level), PC2 (Slope) and PC3 (Curvature) Obtained from the Standard PCA of the Weekly Swap Returns (Right Panel). The similarity of the leading three principal components across all economies indicates the existence of latent global factors.

(continued)
EXHIBIT 7 (continued)
Weekly Swap Rates* for the Eight Economies Considered, with the Maturities of {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 25, 30} Years (Respectively Color-Coded from Blue to Red) for the Period 2015-01-01 to 2019-07-01 (Left Panel) and Their Corresponding PC1 (Level), PC2 (Slope) and PC3 (Curvature) Obtained from the Standard PCA of the Weekly Swap Returns (Right Panel). The similarity of the leading three principal components across all economies indicates the existence of latent global factors.

**Domestic Analysis**

As a complementary and preliminary assessment of the commonality of returns within different country IRS curves, we first performed principal component analysis for each of the eight economies considered, to obtain their dominant domestic principal components, that is, their domestic level, slope, and curvature factors. The three leading factors within each domestic IRS curve are shown in Exhibit 7, and the percentage of variance explained by each component is given in Exhibit 6.

In agreement with the existing literature, here too all economies exhibited similar leading principal components. Moreover, the explanatory power of each component was consistent across all economies, whereby the first principal component (level) explains ≈ 90%, the second principal component (slope) explains ≈ 5%, and the third principal component (curvature) explains ≈ 1% of the total variation in interest rate changes. This supports our proposed multilinear approach to global fixed income returns analysis, as it serves as an indication of the existence of three leading dominant factors, that is, the *global level*, *global slope*, and *global curvature* factors.
With these preliminary and intuitive results, we now proceed to evaluate the common global risk factors using the proposed multilinear model.

**Global Analysis**

We now proceed to evaluate the performance of the proposed multilinear factor analysis when applied to the international IRS dataset. The implementation procedure is summarized as follows:\(^2\)

1. The weekly IRS returns at the t-th week were arranged to form the matrix-valued sample, \( X_t \in \mathbb{R}^{I \times M \times C} \), as described in Equation 1.
2. We estimated the parameters of the model \( (\sigma^2, \Theta^{(m)}, \Theta^{(c)}) \) using the analytic estimators in Equations 3–6.
3. We obtained the global maturity-domain and country-domain factors, \( U^{(m)} \) and \( U^{(c)} \), and their associated eigenvalues, \( \Lambda^{(m)} \) and \( \Lambda^{(c)} \), from the eigendecompositions of \( \Theta^{(m)} \) and \( \Theta^{(c)} \), as shown in Equations 7–8.

The three leading maturity-domain factors, \( \{u^{(m)}_i\}_{i=1}^3 \), are plotted in Exhibit 8a, with their corresponding explanatory powers, \( \{\lambda^{(m)}_i\}_{i=1}^3 \), presented in Exhibit 9. Observe that the maturity-domain factors resemble the components obtained from traditional domestic principal components (see Exhibit 7), and therefore can be interpreted analogously, thus confirming the existence of a common set of bases shared by all economies. Furthermore, the explanatory powers of these factors are on par with those observed from the domestic analyses, which further validates our findings.

Remark 5. The maturity-domain factors obtained from the multilinear analysis, shown in Exhibit 8a, clearly serve as a stencil for describing the term structure within each domestic IRS curve, and as such we refer to these as the global level, global slope, and global curvature.

The country-domain factors, \( \{u^{(c)}_i\}_{i=1}^8 \), are visualized in Exhibits 8a–8c, and their corresponding explanatory powers, \( \{\lambda^{(c)}_i\}_{i=1}^8 \), are presented in Exhibit 10. The most dominant factor, \( u^{(c)}_6 \), had positive entries across all economies, and can be thought of as the global risk premium, analogous to the level factor in the maturity-domain. Notice that this factor also explains a significant portion of the international IRS variance. The remaining country-domain factors represent interpretable macroeconomic factors concerning subsets of the considered economies. These results demonstrate the ability of the proposed approach to yield enhanced

\(^{2}\) Data analysis was implemented using our own Python Higher-Order Tensor ToolBOX (HOTTBOX) (Kisil et al.).
physical insight into the global macroeconomic environment. This is achieved in a straightforward and compact manner, owing to the small number of parameters required to fully describe the global fixed income universe.

Observe that the maturity-domain (listed in Exhibit 9) and country-domain (listed in Exhibit 10) factors also can be directly employed for global macroeconomic hedging and risk management, as elaborated in the Global Portfolio Management and Hedging Section above.

Practical advantages of the proposed multilinear framework therefore include:

- A reduction in the number of parameters required to optimize the global portfolio, which for \( l_m = 15 \) maturities and \( l_c = 8 \) countries reduces from \( l_m l_c = 120 \) to \( (l_m + l_c) = 23 \) portfolio weight parameters, and
- A parsimonious description of the global risk in terms of a multilinear decomposition into the smaller-scale maturity-domain and country-domain risk factors that can facilitate the investor's decision-making process.

EXHIBIT 9
Economic Interpretation and Explanatory Powers of the Three Leading Global Factors in the Maturity-domain

<table>
<thead>
<tr>
<th>Factor</th>
<th>Symbol</th>
<th>Economic Interpretation</th>
<th>Variance Explained (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( u_1 )</td>
<td>Global level</td>
<td>92.37</td>
</tr>
<tr>
<td>2</td>
<td>( u_2 )</td>
<td>Global slope</td>
<td>5.90</td>
</tr>
<tr>
<td>3</td>
<td>( u_3 )</td>
<td>Global curvature</td>
<td>0.97</td>
</tr>
</tbody>
</table>

EXHIBIT 10
Economic Interpretation and Explanatory Powers of the Eight Leading Global Factors in the Country-domain

<table>
<thead>
<tr>
<th>Factor</th>
<th>Symbol</th>
<th>Economic Interpretation</th>
<th>Variance Explained (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( u_1 )</td>
<td>Global risk premium</td>
<td>76.54</td>
</tr>
<tr>
<td>2</td>
<td>( u_2 )</td>
<td>(CA, US) vs. rest</td>
<td>7.93</td>
</tr>
<tr>
<td>3</td>
<td>( u_3 )</td>
<td>(SF, EU, GB) vs. (AU, NZ, US)</td>
<td>7.34</td>
</tr>
<tr>
<td>4</td>
<td>( u_4 )</td>
<td>(GB, AU) vs. rest</td>
<td>3.62</td>
</tr>
<tr>
<td>5</td>
<td>( u_5 )</td>
<td>(GB, NZ, US) vs. rest</td>
<td>2.39</td>
</tr>
<tr>
<td>6</td>
<td>( u_6 )</td>
<td>(GB, NZ, CA) vs. rest</td>
<td>1.55</td>
</tr>
<tr>
<td>7</td>
<td>( u_7 )</td>
<td>JPY vs. rest</td>
<td>0.43</td>
</tr>
<tr>
<td>8</td>
<td>( u_8 )</td>
<td>SF vs. EU</td>
<td>0.21</td>
</tr>
</tbody>
</table>

CONCLUSIONS

We introduce a novel multilinear algebraic framework for modeling the covariance of global fixed income returns in order to cater for the highly structured correlations across maturities and economies. By virtue of the multilinear approach inherent to tensors, as opposed to the standard “flat-view” multivariate matrix ones based on linear algebra, we show the covariance structure of international fixed income returns to be factorizable analytically into the maturity-domain and country-domain covariances. In this way, the proposed analysis: 1) achieves a significant reduction in the number of parameters required to fully describe the international investment universe; and 2) offers a mathematically tractable platform for estimating and identifying physically interpretable global risk factors.

As a natural extension of the proposed framework, we have derived analytic solutions to the global hedging and portfolio management paradigms, which allow the investor to gain enhanced control over the portfolio risk within the two independent domains spanning the global returns landscape: maturity and country. We have performed empirical analysis on the interest rate swaps curves for eight developed economies, and confirmed the existence of common global risk factors and their practical utility. The results are supported by our own Python toolbox for multilinear tensor analysis (Kisil et al.).

REFERENCES


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